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AN ITERATIVE METHOD FOR COMMON SOLUTION TO VARIOUS PROBLEMS

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Abstract. In this paper, we introduce a new iterative method to find a common solution of a generalized mixed equilibrium problem, a variational inequality problem and a hierarchical fixed point problem for demicontinuous nearly nonexpansive mappings. We prove that our method converges strongly to a common solution of all above problems. It is worth noting that Main Theorem is proved without usual demiclosedness condition. As our iterative method generalizes several methods, the results here improve and extend many recent results.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, C a nonempty closed convex subset of H. Let $\mathbb{R}$ denote the set of all real numbers, $G : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ a function and B a nonlinear mapping. The generalized mixed equilibrium problem ($GMEP$), is finding a point $x \in C$ such that

$$G(x, y) + \varphi(y) - \varphi(x) + \langle Bx, y - x \rangle \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions of the problem (1.1) is denoted by $GMEP(G, \varphi, B)$. In the problem (1.1), if we take $B = 0$, then it is reduced to the mixed equilibrium problem, denoted by MEP, which is to find a point $x \in C$ such that

$$G(x, y) + \varphi(y) - \varphi(x) \geq 0, \forall y \in C.$$

The set of solutions of the mixed equilibrium problem is denoted by $MEP(G, \varphi)$. In case $\varphi = 0$ in the problem (1.1), it is reduced to generalized equilibrium problem, denoted by GEP of finding a point $x \in C$ such that

$$G(x, y) + \langle Bx, y - x \rangle \geq 0, \ \forall y \in C$$

whose set of solutions is denoted by $GEP(G, B)$. If we take $\varphi = 0$ and $G(x, y) = 0$ for all $x, y \in C$, then it is equivalent to find a $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \ \forall y \in C \tag{1.2}$$

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which is called the variational inequality problem, denoted by $VI(C,B)$. The solution of $VI(C,B)$ is denoted by Ω, i.e.,

$$\Omega = \{x \in C : \langle Bx, y - x\rangle \geq 0,\ \forall y \in C\}.$$

The generalized mixed equilibrium problem is very general in the sense that it includes, as special cases, the optimization problem, the variational inequality problem, the fixed point problem, the nonlinear complementarity problem, the Nash equilibrium problem in noncooperative games, the vector optimization problem, the saddle point problem, the minimization problem and so forth. Hence, some solution methods have been studied to solve generalized mixed equilibrium problem by many authors; see, for example [2, 8, 10].

On the other hand, we consider another problem that is called hierarchical fixed point problem. Let $S : C \to H$ be a mapping. To hierarchically find a fixed point of a mapping T with respect to another mapping S is to find an $x^* \in Fix(T)$ such that

$$\langle x^* - Sx^*, x - x^*\rangle \geq 0,\ \ x \in Fix(T), \tag{1.3}$$

where $Fix(T)$ is the set of fixed points of T, i.e., $Fix(T) = \{x \in C : Tx = x\}$. It is known that the hierarchical fixed point problem (1.3) is related with some monotone variational inequalities and convex programming problems; see for example [4, 16]. Hence, various methods to solve a hierarchical fixed point problem have been studied by many authors; see, for example [6, 16] and the references therein.

Now, we give some definitions of nonlinear mappings which are used in the next sections. Let $T : C \to H$ be a mapping. If there exits a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$, then T is called L-Lipschitzian. In particular, if $L \in (0, 1)$, then T is said to be a contraction; if $L = 1$, then T is called a nonexpansive mapping. Fix a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$. If $\|T^n x - T^n y\| \leq \|x - y\| + a_n$ holds for all $x, y \in C$ and $n \geq 1$, then T is said to be nearly nonexpansive with respect to $\{a_n\}$ [9]. It is clear that the class of nearly nonexpansive mappings is a wider class of nonexpansive mappings. A mapping $F : C \to H$ is called η-strongly monotone operator if there exists a constant $\eta \geq 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2,\ \forall x, y \in C.$$

In particular, if $\eta = 0$, then F is said to be monotone.

Below, we gather some iterative processes which are related with the problems (1.1), (1.2) and (1.3).

Ceng et al. [3] proved that the sequence $\{x_n\}$ generated by the iterative method:

$$x_{n+1} = P_C[\alpha_n\rho V x_n + (1 - \alpha_n\mu F)Tx_n], \forall n \geq 1, \tag{1.4}$$

converges strongly to the unique solution of the variational inequality

$$\langle(\rho V - \mu F)x^*, x - x^*\rangle \leq 0,\ \forall x \in Fix(T). \tag{1.5}$$

Recently, Wang and Xu [13] proved that the more general sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n) x_n, \\ x_{n+1} = P_C [\alpha_n \rho V x_n + (I - \alpha_n \mu F) T y_n], \ \forall n \geq 1. \end{cases} \tag{1.6}$$

converges strongly to the hierarchical fixed point of T with respect to the mapping S which is the unique solution of the variational inequality (1.5).

More recently, Bnouhachem and Noor [1] introduced the following iterative scheme

$$\begin{cases} G(u_n, y) + \langle Bx, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C \\ z_n = P_C (u_n - \lambda_n A u_n), \\ y_n = P_C (\beta_n S x_n + (1-\beta_n) z_n), \\ x_{n+1} = P_C (\alpha_n f x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n), \forall n \geq 1. \end{cases}, \tag{1.7}$$

They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $z \in P_{\Omega \cap GEP(G,B) \cap Fix(T)} f(z)$, a common solution of a variational inequality problem, a generalized equilibrium problem and a hierarchical fixed point problem. z is the unique solution of the following variational inequality:

$$\langle (I - f) z, x - z \rangle \geq 0, \forall x \in \Omega \cap GEP(G) \cap Fix(T),$$

where $Fix(T) = \bigcap_{i=1}^{\infty} Fix(T_i)$.

In this paper, motivated and inspired by the above iterative methods, we introduce an iterative projection method. We prove a strong convergence theorem to compute an approximate element of the common set of solutions of a generalized mixed equilibrium problem, a variational inequality problem and a hierarchically fixed point problem for nearly nonexpansive mappings. Our method generalizes many known results; for example, Yao et. al. [16], Marino and Xu [6], Ceng et. al. [3], Wang and Xu [13], Moudafi [7], Xu [14], Tian [12] and Suzuki [11] and references therein.

2. Preliminaries

In this section, we gather some useful lemmas and definitions which we need for the next section. Throughout this paper, we use " $\rightarrow$ " and " $\rightharpoonup$ " for the strong and weak convergence, respectively. Let C be a nonempty closed convex subset of a real Hilbert space H. It is known that for any $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that $\|x - P_C x\| = \inf_{y \in C} \|x - y\|, \forall x \in H$.

The mapping $P_C : H \rightarrow C$ is called a metric projection. For a metric projection P_C, the following inequalities are hold:

(1) $\|P_C x - P_C y\| \leq \|x - y\|, \forall x, y \in H$,
(2) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \forall x, y \in H$,
(3) $\langle x - P_C x, P_C x - y \rangle \geq 0, \forall x \in H, y \in C$.

Lemma 1. *[3] Let $V : C \to H$ be a γ-Lipschitzian mapping with a constant $\gamma \geq 0$ and let $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator with constants $L, \eta > 0$. Then for $0 \leq \rho\gamma < \mu\eta$,*

$$\langle (\mu F - \rho V)\, x - (\mu F - \rho V)\, y, x - y \rangle \geq (\mu\eta - \rho\gamma) \left\| x - y \right\|^2, \ \forall x, y \in C.$$

That is, $\mu F - \rho V$ is strongly monotone with coefficient $\mu\eta - \rho\gamma$.

Lemma 2. *[15] Let C be a nonempty subset of a real Hilbert space H. Suppose that $\lambda \in (0, 1)$ and $\mu > 0$. Let $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator on C. Then the mapping $g : C \to H$ defined by $g := I - \lambda\mu F$ satisfies for $\mu \in \left(0, \frac{2\eta}{L^2}\right)$,*

$$\left\| g(x) - g(y) \right\| \leq (1 - \lambda\nu) \left\| x - y \right\|, \ \forall x, y \in C,$$

where $\nu = 1 - \sqrt{1 - \mu\,(2\eta - \mu L^2)}$.

Lemma 3. *[14] Assume that $\{x_n\}$ is a sequence of nonnegative real numbers satisfying the conditions $x_{n+1} \leq (1 - \alpha_n)\, x_n + \alpha_n \beta_n$, $\forall n \geq 1$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \beta_n \leq 0$. Then $\lim_{n\to\infty} x_n = 0$.*

For solving an equilibrium problem for a bifunction $G : C \times C \to \mathbb{R}$, let us assume that G satisfies the following conditions:

(A1) $G(x, x) = 0$, $\forall x \in C$,
(A2) G is monotone, i.e. $G(x, y) + G(y, x) \leq 0$, $\forall x, y \in C$,
(A3) $\forall x, y, z \in C, \lim_{t\to 0^+} G(tz + (1 - t)\, x, y) \leq G(x, y)$,
(A4) $\forall x \in C$, $y \longmapsto G(x, y)$ is convex and lower semicontinuous,
(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$

$$G(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0,$$

(B2) C is a bounded set.

Lemma 4. *[8] Let C be a nonempty closed convex subset of a Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) := \left\{ z \in C : G(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(1) *For each $x \in H$, $T_r(x) \neq \emptyset$*
(2) *T_r is single valued*
(3) *T_r is firmly nonexpansive i.e. $\left\| T_r x - T_r y \right\|^2 \leq \langle T_r x - T_r y, x - y \rangle$, $\forall x, y \in H$*
(4) *$Fix(T_r) = MEP(G)$*

(5) $MEP(G)$ *is closed and convex.*

Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mapping. For a sequence $\{x_n\}$ in C which converges strongly to $x \in X$, if $\{Tx_n\}$ converges weakly to Tx, then T is called demicontinuous.

Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $\{x_n\}$ be a bounded sequence in X and $r : C \to [0, \infty)$ be a functional defined by $r(x) = \limsup_{n\to\infty} \|x_n - x\|$, $x \in C$. The infimum of $r(\cdot)$ over C is called asymptotic radius of $\{x_n\}$ with respect to C and it is denoted by $r(C, \{x_n\})$. A point $w \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to C if $r(w) = \min\{r(x) : x \in C\}$. The set of all asymptotic centers is denoted by $A(C, \{x_n\})$. Following is a well-known result.

Lemma 5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X satisfying the Opial condition. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup w$, then w is the asymptotic center of $\{x_n\}$ in C.*

Lemma 6. *[9] Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be a demicontinuous nearly Lipschitzian mapping with sequence $\{a_n, \eta(T^n)\}$ such that* $\lim_{n\to\infty} \eta(T^n) \le 1$. *If $\{x_n\}$ is a bounded sequence in C such that*

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\|x_n - T^m x_n\|\right) = 0 \text{ and } A(C, \{x_n\}) = \{w\},$$

then w is a fixed point of T.

3. Main Results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A, B : C \to H$ be α, θ-inverse strongly monotone mappings, respectively. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (A1)-(A4), $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, $S : C \to H$ be a nonexpansive and T be a demicontinuous nearly nonexpansive mapping with the sequence $\{a_n\}$ such that $\mathcal{F} := Fix(T) \cap \Omega \cap GMEP(G, \varphi, B) \neq \emptyset$. Let $V : C \to H$ be a γ-Lipschitzian mapping, $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator such that these coefficients satisfy $0 < \mu < \frac{2\eta}{L^2}$, $0 \le \rho\gamma < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. Assume that either (B1) or (B2) holds. For an arbitrarily chosen initial value x_1, define the sequence $\{x_n\}$ in C generated by

$$\begin{cases} G(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \forall y \in C \\ z_n = P_C(u_n - \lambda_n A u_n), \\ y_n = P_C[\beta_n S x_n + (1 - \beta_n) z_n], \\ x_{n+1} = P_C[\alpha_n \rho V x_n + (I - \alpha_n \mu F) T^n y_n], \ n \ge 1, \end{cases} \tag{3.1}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{r_n\} \subset (0, 2\theta)$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

It is known that convergence of a sequence depends on the choice of the control sequences and mappings. So, we consider the following hypotheses on our control sequences and mappings:

$(C1)$ $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$;

$(C2)$ $\lim_{n\to\infty}\frac{a_n}{\alpha_n} = 0,\ \lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = 0,\ \lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n} = 0,\ \lim_{n\to\infty}\frac{|\lambda_n-\lambda_{n-1}|}{\alpha_n} = 0$

$\lim_{n\to\infty}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n} = 0$, and $\lim_{n\to\infty}\frac{|r_n-r_{n-1}|}{\alpha_n} = 0$;

$(C3)$ $\lim_{n\to\infty}\left\|T^n x - T^{n-1}x\right\| = 0$ and $\lim_{n\to\infty}\frac{\left\|T^n x - T^{n-1}x\right\|}{\alpha_n} = 0, \forall x\in C.$

Before giving the main theorem, we have to prove the following lemmas.

Lemma 7. *Assume that the conditions* $(C1)$ *and* $(C2)$ *hold. Let* $p\in\mathcal{F}$*. Then, the sequences* $\{x_n\},\{y_n\},\{z_n\}$ *and* $\{u_n\}$ *generated by (3.1) are bounded.*

Proof. It is easy to see that the mappings $I-r_nB$ and $I-\lambda_nA$ are nonexpansive. From Lemma 4, we have $u_n = T_{r_n}(x_n - r_nBx_n)$. Let $p\in\mathcal{F}$. So, we get $p = T_{r_n}(p - r_nBp)$ and

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{r_n}(x_n-r_nBx_n) - T_{r_n}(p-r_nBp)\|^2\\ &\le \|(x_n-r_nBx_n)-(p-r_nBp)\|^2\\ &\le \|x_n-p\|^2 - r_n(2\theta-r_n)\|Bx_n-Bp\|^2\\ &\le \|x_n-p\|^2. \end{aligned}\tag{3.2}$$

By using (3.2), we obtain

$$\begin{aligned}\|z_n-p\|^2 &= \|P_C(u_n-\lambda_nAu_n) - P_C(p-\lambda_nAp)\|^2\\ &\le \|u_n-p\|^2 - \lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\\ &\le \|x_n-p\|^2. \end{aligned}\tag{3.3}$$

So, from (3.3), we have

$$\begin{aligned}\|y_n-p\| &= \|P_C[\beta_nSx_n + (1-\beta_n)x_n] - P_Cp\|\\ &\le \|x_n-p\| + \beta_n\|Sp-p\|. \end{aligned}\tag{3.4}$$

Since $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = 0$, without loss of generality, we can assume that $\beta_n \le \alpha_n$, for all $n\ge 1$. This gives us $\lim_{n\to\infty}\beta_n = 0$. Let $t_n = \alpha_n\rho Vx_n + (I-\alpha_n\mu F)T^ny_n$. Then, we have

$$\begin{aligned}\|x_{n+1}-p\| &\le \|\alpha_n\rho Vx_n + (I-\alpha_n\mu F)T^ny_n - p\|\\ &\le \alpha_n\rho\gamma\|x_n-p\| + \alpha_n\|\rho Vp-\mu Fp\|\\ &\quad + (1-\alpha_n\nu)(\|y_n-p\| + a_n). \end{aligned}\tag{3.5}$$

So, it follows from (3.4) and (3.5) that

$$\begin{aligned}\|x_{n+1}-p\| &\leq \alpha_n\rho\gamma\|x_n-p\|+\alpha_n\|\rho Vp-\mu Fp\| \\ &\quad +(1-\alpha_n\nu)(\|x_n-p\|+\beta_n\|Sp-p\|+a_n) \\ &\leq (1-\alpha_n(\nu-\rho\gamma))\|x_n-p\| \\ &\quad +\alpha_n(\nu-\rho\gamma)\left[\frac{1}{(\nu-\rho\gamma)}\left(\|\rho Vp-\mu Fp\|+\|Sp-p\|+\frac{a_n}{\alpha_n}\right)\right] \quad (3.6)\end{aligned}$$

From condition $(C2)$, there exists a constant $M_1>0$ such that

$$\|\rho Vp-\mu Fp\|+\|Sp-p\|+\frac{a_n}{\alpha_n}\leq M_1,\ \forall n\geq 1.$$

Hence, from (3.6) we get

$$\|x_{n+1}-p\|\leq(1-\alpha_n(\nu-\rho\gamma))\|x_n-p\|+\alpha_n(\nu-\rho\gamma)\frac{M_1}{(\nu-\rho\gamma)}.$$

By induction, we obtain

$$\|x_{n+1}-p\|\leq\max\left\{\|x_1-p\|,\frac{M}{(\nu-\rho\gamma)}\right\}.$$

Therefore, we obtain that $\{x_n\}$ is bounded. So the sequences $\{y_n\},\{z_n\}$ and $\{u_n\}$ are also bounded. □

Lemma 8. *Assume that* $(C1)$-$(C3)$ *hold and* $p\in\mathcal{F}$. *Let* $\{x_n\}$ *be the sequence generated by (3.1). Then, the followings are true:*

(i) $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.
(ii) *The weak w-limit set* $w_w(x_n)\subset Fix(T)$.

Proof. (i) Since P_C and $(I-\lambda_nA)$ are nonexpansive mappings, we have

$$\begin{aligned}\|z_n-z_{n-1}\| &\leq \|(u_n-\lambda_nAu_n)-(u_{n-1}-\lambda_{n-1}Au_{n-1})\| \\ &= \|u_n-u_{n-1}-\lambda_n(Au_n-Au_{n-1})-(\lambda_n-\lambda_{n-1})Au_{n-1}\| \\ &\leq \|u_n-u_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Au_{n-1}\|, \quad (3.7)\end{aligned}$$

and so

$$\begin{aligned}\|y_n-y_{n-1}\| &= \|P_C[\beta_nSx_n+(1-\beta_n)z_n] \\ &\quad -P_C[\beta_{n-1}Sx_{n-1}-(1-\beta_{n-1})z_{n-1}]\| \\ &\leq \beta_n\|x_n-x_{n-1}\|+(1-\beta_n)[\|u_n-u_{n-1}\| \\ &\quad +|\lambda_n-\lambda_{n-1}|\|Au_{n-1}\|] \\ &\quad +|\beta_n-\beta_{n-1}|(\|Sx_{n-1}\|+\|z_{n-1}\|). \quad (3.8)\end{aligned}$$

On the other side, since $u_n=T_{r_n}(x_n-r_nBx_n)$ and $u_{n-1}=T_{r_{n-1}}(x_{n-1}-r_{n-1}Bx_{n-1})$, we get

$$\begin{aligned}&G(u_n,y)+\varphi(y)-\varphi(u_n)+\langle Bx_n,y-u_n\rangle \\ &+\tfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\ \forall y\in C\end{aligned} \quad (3.9)$$

and

$$\begin{array}{l} G\left(u_{n-1}, y\right)+\varphi(y)-\varphi\left(u_{n-1}\right)+\left\langle B x_{n-1}, y-u_{n-1}\right\rangle \\ +\frac{1}{r_{n-1}}\left\langle y-u_{n-1}, u_{n-1}-x_{n-1}\right\rangle \geq 0, \forall y \in C. \end{array} \tag{3.10}$$

If we take $y=u_{n-1}$ in (3.9) and $y=u_n$ in (3.10), then we have

$$\begin{array}{l} G\left(u_n, u_{n-1}\right)+\varphi\left(u_{n-1}\right)-\varphi\left(u_n\right)+\left\langle B x_n, u_{n-1}-u_n\right\rangle \\ +\frac{1}{r_n}\left\langle u_{n-1}-u_n, u_n-x_n\right\rangle \geq 0, \end{array} \tag{3.11}$$

and

$$\begin{array}{l} G\left(u_{n-1}, u_n\right)+\varphi\left(u_n\right)-\varphi\left(u_{n-1}\right)+\left\langle B x_{n-1}, u_n-u_{n-1}\right\rangle \\ +\frac{1}{r_{n-1}}\left\langle u_n-u_{n-1}, u_{n-1}-x_{n-1}\right\rangle \geq 0. \end{array} \tag{3.12}$$

By using the monotonicity of the bifunction G and the inequalitites (3.11) and (3.12), we obtain

$$\left\langle B x_{n-1}-B x_n, u_n-u_{n-1}\right\rangle+\left\langle u_n-u_{n-1}, \frac{u_{n-1}-x_{n-1}}{r_{n-1}}-\frac{u_n-x_n}{r_n}\right\rangle \geq 0.$$

It follows from the last inequality that

$$\left\|u_{n-1}-u_n\right\| \leq\left|1-\frac{r_n}{r_{n-1}}\right|\left\|u_{n-1}-x_{n-1}\right\|+\left\|x_{n-1}-x_n\right\| .$$

Without loss of generality, we can assume that there exists a real number μ such that $r_n>\mu>0$ for all positive integers n. Then, we have

$$\left\|u_{n-1}-u_n\right\| \leq\left\|x_{n-1}-x_n\right\|+\frac{1}{\mu}\left|r_{n-1}-r_n\right|\left\|u_{n-1}-x_{n-1}\right\| . \tag{3.13}$$

(3.8) and (3.13) imply that

$$\begin{aligned} \left\|y_n-y_{n-1}\right\| \leq\; & \beta_n\left\|x_n-x_{n-1}\right\| \\ & +\left(1-\beta_n\right)\left[\left\|x_{n-1}-x_n\right\|+\frac{1}{\mu}\left|r_{n-1}-r_n\right|\left\|u_{n-1}-x_{n-1}\right\|\right. \\ & \left.+\left|\lambda_n-\lambda_{n-1}\right|\left\|A u_{n-1}\right\|\right]+\left|\beta_n-\beta_{n-1}\right|\left(\left\|S x_{n-1}\right\|+\left\|z_{n-1}\right\|\right) \\ =\; & \left\|x_n-x_{n-1}\right\|+\left(1-\beta_n\right)\left[\frac{1}{\mu}\left|r_{n-1}-r_n\right|\left\|u_{n-1}-x_{n-1}\right\|\right. \\ & \left.+\left|\lambda_n-\lambda_{n-1}\right|\left\|A u_{n-1}\right\|\right]+\left|\beta_n-\beta_{n-1}\right|\left(\left\|S x_{n-1}\right\|+\left\|z_{n-1}\right\|\right) . \end{aligned}$$

Hence, we get

$$\begin{aligned} \left\|x_{n+1}-x_n\right\| \leq\; & \left(1-\alpha_n(\nu-\rho \gamma)\right)\left\|x_n-x_{n-1}\right\|+\left(1+\mu \alpha_{n-1} L\right)\left\|T^n y_{n-1}-T^{n-1} y_{n-1}\right\| \\ & +M_2\left(\left|\alpha_n-\alpha_{n-1}\right|+\frac{1}{\mu}\left|r_{n-1}-r_n\right|\right. \\ & \left.+\left|\lambda_n-\lambda_{n-1}\right|+\left|\beta_n-\beta_{n-1}\right|\right)+a_n \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} M_2 &= \max\left\{\sup_{n\geq 1}\left(\gamma\left\|Vx_{n-1}\right\|+\left\|FT^n y_{n-1}\right\|\right), \sup_{n\geq 1}\left\|u_{n-1}-x_{n-1}\right\|,\right. \\ &\left. \sup_{n\geq 1}\left\|Au_{n-1}\right\|, \sup_{n\geq 1}\left(\left\|Sx_{n-1}\right\|+\left\|z_{n-1}\right\|\right)\right\}. \end{aligned}$$

Therefore, we obtain

$$\|x_{n+1}-x_n\| \leq (1-\alpha_n(\nu-\rho\gamma))\|x_n-x_{n-1}\|+\alpha_n(\nu-\rho\gamma)\delta_n, \quad (3.15)$$

where

$$\begin{aligned} \delta_n &= \frac{1}{(\nu-\rho\gamma)}\left[(1+\mu\alpha_{n-1}L)\frac{\left\|T^n y_{n-1}-T^{n-1}y_{n-1}\right\|}{\alpha_n}\right. \\ &+\frac{a_n}{\alpha_n}+M_2\left(\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\frac{1}{\mu}\frac{|r_{n-1}-r_n|}{\alpha_n}\right. \\ &\left.\left.+\frac{|\lambda_n-\lambda_{n-1}|}{\alpha_n}+\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}\right)\right]. \end{aligned}$$

By using conditions $(C2)$ and $(C3)$, we get $\limsup_{n\to\infty}\delta_n \leq 0$ and hence from (3.15) and Lemma 3, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \quad (3.16)$$

(ii) Now, we show that the weak w-limit set of $\{x_n\}$ is a subset of the set of fixed points of T. To show that, we need to show $\lim_{n\to\infty}\|u_n-x_n\| = 0$. Let $p \in \mathcal{F}$. Then, by using (3.2) and (3.3), we get

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \|t_n-p\|^2 \\ &\leq \alpha_n\|\rho Vx_n-\mu Fp\|^2+\beta_n\|Sx_n-p\|^2+\|x_n-p\|^2 \\ &-(1-\alpha_n\nu)(1-\beta_n)\left[r_n(2\theta-r_n)\|Bx_n-Bp\|^2\right. \\ &\left.+\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\right] \\ &+2(1-\alpha_n\nu)a_n\|y_n-p\|+(1-\alpha_n\nu)a_n^2. \quad (3.17) \end{aligned}$$

The inequality (3.17) implies that

$$\begin{aligned} &(1-\alpha_n\nu)(1-\beta_n)\left\{r_n(2\theta-r_n)\|Bx_n-Bp\|^2+\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\right\} \\ \leq\ &\alpha_n\|\rho Vx_n-\mu Fp\|^2+\beta_n\|Sx_n-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-p\| \\ &+2(1-\alpha_n\nu)a_n\|y_n-p\|+(1-\alpha_n\nu)a_n^2. \end{aligned}$$

A joint effect of (3.16), conditions $(C1)$ and $(C2)$ yields that $\lim_{n\to\infty}\|Bx_n-Bp\| = 0$ and $\lim_{n\to\infty}\|Au_n-Ap\| = 0$. On the other side, we know from Lemma 4 that T_{r_n} is firmly nonexpansive mapping, thus we get

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(p - r_n Bp)\|^2 \\ &\le \langle u_n - p, (x_n - r_n Bx_n) - (p - r_n Bp)\rangle \\ &= \frac{1}{2}\Big\{\|u_n - p\|^2 + \|(x_n - r_n Bx_n) - (p - r_n Bp)\|^2 \\ &\quad - \|u_n - p - [(x_n - r_n Bx_n) - (p - r_n Bp)]\|^2\Big\}\end{aligned}$$

which implies that

$$\begin{aligned}\|u_n - p\|^2 &\le \|x_n - p\|^2 - \|u_n - x_n\|^2 \\ &\quad + 2r_n \|u_n - x_n\| \|Bx_n - Bp\|. \end{aligned} \tag{3.18}$$

Then, from (3.3), (3.17) and (3.18), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n \|\rho V x_n - \mu F p\|^2 + \beta_n \|Sx_n - p\|^2 + \|x_n - p\|^2 \\ &\quad - (1 - \alpha_n \nu)(1 - \beta_n)\|u_n - x_n\|^2 + 2r_n \|u_n - x_n\| \|Bx_n - Bp\| \\ &\quad + 2(1 - \alpha_n \nu) a_n \|y_n - p\| + (1 - \alpha_n \nu) a_n^2,\end{aligned}$$

and so

$$\begin{aligned}&(1 - \alpha_n \nu)(1 - \beta_n)\|u_n - x_n\|^2 \\ \le\ & \alpha_n \|\rho V x_n - \mu F p\|^2 + \beta_n \|Sx_n - p\|^2 \\ & + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| \\ & + 2r_n \|u_n - x_n\| \|Bx_n - Bp\| \\ & + 2(1 - \alpha_n \nu) a_n \|y_n - p\| + (1 - \alpha_n \nu) a_n^2.\end{aligned}$$

Since $\lim_{n\to\infty} \|Bx_n - Bp\| = 0$ and $\{\|y_n - p\|\}$ is a bounded sequence, it follows from (3.16) and conditions $(C1)$, $(C2)$ that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.19}$$

On the other hand, from the property (2) of the metric projection, we can write

$$\begin{aligned}\|z_n - p\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(p - \lambda_n Ap)\|^2 \\ &\le \frac{1}{2}\Big\{\|z_n - p\|^2 + \|u_n - p\|^2 - \|u_n - z_n\|^2 \\ &\quad + 2\lambda_n \|u_n - z_n\| \|Au_n - Ap\|\Big\}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|z_n - p\|^2 &\le \|x_n - p\|^2 - \|u_n - z_n\|^2 \\ &\quad + 2\lambda_n \|u_n - z_n\| \|Au_n - Ap\|. \end{aligned} \tag{3.20}$$

From (3.17) and (3.20), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n \|\rho V x_n - \mu F p\|^2 + \beta_n \|Sx_n - p\|^2 + \|x_n - p\|^2 \\ &\quad - (1 - \alpha_n \nu)\beta_n \|u_n - z_n\|^2 + 2\lambda_n \|u_n - z_n\| \|Au_n - Ap\| \\ &\quad + 2(1 - \alpha_n \nu) a_n \|y_n - p\| + (1 - \alpha_n \nu) a_n^2.\end{aligned}$$

So, we have

$$\begin{aligned}(1-\alpha_n\nu)\beta_n\|u_n-z_n\|^2 &\leq \alpha_n\|\rho Vx_n-\mu Fp\|^2+\beta_n\|Sx_n-p\|^2\\ &+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|\\ &+2\lambda_n\|u_n-z_n\|\|Au_n-Ap\|\\ &+2(1-\alpha_n\nu)a_n\|y_n-p\|+(1-\alpha_n\nu)a_n^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|Au_n-Ap\|=0$ and $\{\|y_n-p\|\}$ is a bounded sequence, it follows from (3.16) and conditions $(C1)$ and $(C2)$ that $\lim_{n\to\infty}\|u_n-z_n\|=0$. Now by using (3.19) we get

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.21}$$

Moreover,

$$\begin{aligned}\|x_n-y_n\| &\leq \|x_n-u_n\|+\|u_n-z_n\|+\|z_n-y_n\|\\ &= \|x_n-u_n\|+\|u_n-z_n\|+\beta_n(Sx_n-z_n).\end{aligned}$$

Since $\lim_{n\to\infty}\beta_n=0$, we obtain

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.22}$$

Now, we show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. Before that, we need to show that $\lim_{m\to\infty}(\lim_{n\to\infty}\|x_n-T^mx_n\|)=0$. For $n\geq m\geq 1$,

$$\begin{aligned}\|T^ny_n-T^mx_n\| &\leq \left\|T^ny_n-T^{n-1}y_n\right\|+\left\|T^{n-1}y_n-T^{n-2}y_n\right\|\\ &+\cdots+\|y_n-x_n\|+a_m,\end{aligned} \tag{3.23}$$

and so

$$\begin{aligned}&\|x_{n+1}-T^mx_n\|\\ \leq\ &\alpha_n\|\rho Vx_n-\mu FT^ny_n\|+\left\|T^ny_n-T^{n-1}y_n\right\|\\ &+\left\|T^{n-1}y_n-T^{n-2}y_n\right\|+\cdots+\|y_n-x_n\|+a_m.\end{aligned} \tag{3.24}$$

Thus by (3.23) and (3.24), we are able to say that

$$\begin{aligned}\|x_n-T^mx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T^mx_n\|\\ &\leq \|x_n-x_{n+1}\|+\alpha_n\|\rho Vx_n-\mu FT^ny_n\|\\ &+\left\|T^ny_n-T^{n-1}y_n\right\|+\left\|T^{n-1}y_n-T^{n-2}y_n\right\|\\ &+\cdots+\|y_n-x_n\|+a_m.\end{aligned}$$

Since $\|\rho Vx_n-\mu FT^ny_n\|$ is bounded and $a_n\to 0$, appealing to (3.16), (3.22), conditions $(C1)$ and $(C3)$, we reach at

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\|x_n-T^mx_n\|\right)=0. \tag{3.25}$$

Consequently,

$$\|x_n-Tx_n\|\leq\|x_n-T^mx_n\|+\|T^mx_n-Tx_n\|\to 0 \text{ as } n,m\to\infty.$$

Since $\{x_n\}$ is bounded, there exists a weak convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $x_{n_k}\rightharpoonup w$ as $k\to\infty$. Then, Opial's condition guarantees that the weak subsequential limit of $\{x_n\}$ is unique. Hence $x_n\rightharpoonup w$ as $n\to\infty$. Now bringing (3.25), Theorem 5 and Lemma 6 into play, $w \in Fix(T)$. That is, $w_w(x_n) \subset Fix(T)$. □

Theorem 1. *Assume that* $(C1)$-$(C3)$ *hold. Then the sequence* $\{x_n\}$ *generated by (3.1) converges strongly to* $x^* \in \mathcal{F}$*, which is the unique solution of the variational inequality*

$$\langle (\rho V - \mu F) x^*, x - x^* \rangle \leq 0, \ \forall x \in \mathcal{F}. \tag{3.26}$$

Proof. From Lemma 1, since the operator $\mu F - \rho V$ is $(\mu\eta - \rho\gamma)$-strongly monotone we get the uniqueness of the solution of the variational inequality (3.26). Denote this solution by $x^* \in \mathcal{F}$.

Now, we divide our proof into three steps.

Step 1. From Lemma 7, since $\{x_n\}$ is a bounded sequence, there exists an element w such that $x_n \rightharpoonup w$. Now, we show that $w \in \mathcal{F} = Fix(T) \cap \Omega \cap GMEP(G, \varphi, B)$. Firstly, it follows from Lemma 8 (ii) that $w \in Fix(T)$. Secondly, we show that $w \in GMEP(G, \varphi, B)$. Since $u_n = T_{r_n}(x_n - r_n B x_n)$, we have

$$G(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C$$

which implies that

$$\varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq G(y, u_n), \forall y \in C,$$

and hence

$$\varphi(y) - \varphi(u_{n_k}) + \langle Bx_{n_k}, y - u_{n_k} \rangle + \left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq G(y, u_{n_k}), \forall y \in C. \tag{3.27}$$

Let $y \in C$ and $y_t = ty + (1-t) w$, for $t \in (0, 1]$. Then, $y_t \in C$. From (3.27), we obtain

$$\begin{aligned} \langle By_t, y_t - u_{n_k} \rangle \geq & \ \langle By_t - Bx_{n_k}, y_t - u_{n_k} \rangle + \langle Bu_{n_k} - Bx_{n_k}, y_t - u_{n_k} \rangle - \varphi(y_t) \\ & + \varphi(u_{n_k}) - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle + G(y_t, u_{n_k}). \end{aligned} \tag{3.28}$$

Using (3.19) and Lipschitz continuity of B, we obtain $\lim_{k\to\infty} \|Bu_{n_k} - Bx_{n_k}\| = 0$. So, it follows from (3.28), $u_{n_k} \rightharpoonup w$ and the monotonicity of B that

$$\langle By_t, y_t - w \rangle \geq G(y_t, w) - \varphi(y_t) + \varphi(w). \tag{3.29}$$

By using the inequality (3.29) and assumptions (A1)-(A4), we get

$$\begin{aligned} 0 &= G(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq t [G(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t) t \langle By_t, y - w \rangle. \end{aligned}$$

That is,

$$G(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t) \langle By_t, y - w \rangle \geq 0.$$

If we take limit as $t \to 0^+$ for all $y \in C$, we get

$$G(w, y) + \varphi(y) - \varphi(w) + \langle Bw, y - w \rangle \geq 0, \forall y \in C.$$

Hence, we have $w \in GMEP(G, \varphi, B)$. Finally, we show that $w \in \Omega$. Let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}.$$

Let K be a mapping defined by

$$Kv = \begin{cases} Av + N_C v & , \ v \in C, \\ \emptyset & , \ v \notin C. \end{cases}$$

Then, it is known that K is maximal monotone mapping. Let $(v, u) \in G(K)$. From the definition of the the mapping K, since $u - Av \in N_C v$ and $z_n \in C$, we get

$$\langle v - z_n, u - Av \rangle \geq 0. \tag{3.30}$$

Also, by using the definition of z_n, we get

$$\langle v - z_n, z_n - u_n - \lambda_n A u_n \rangle \geq 0$$

and so,

$$\left\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + A u_n \right\rangle \geq 0.$$

Hence, from (3.30), we obtain

$$\langle v - z_{n_i}, u \rangle \geq \langle v - z_{n_i}, A z_{n_i} - A u_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle.$$

It now follows from (3.21) that $u_{n_i} \rightharpoonup w$ and $z_{n_i} \rightharpoonup w$ for $i \to \infty$ and so

$$\langle v - w, u \rangle \geq 0.$$

Since K is maximal monotone, we have $w \in K^{-1}0$ and hence $w \in \Omega$. Thus, we obtain $w \in \mathcal{F} = Fix(T) \cap \Omega \cap GMEP(G)$.

Step 2. In this step, we show that $\limsup_{n\to\infty} \langle (\rho V - \mu F) x^*, x_n - x^* \rangle \leq 0$, where x^* is the unique solution of the variational inequality (3.26). Since the sequence $\{x_n\}$ is bounded, it has a weak convergent subsequence $\{x_{n_k}\}$ such that

$$\limsup_{n\to\infty} \langle (\rho V - \mu F) x^*, x_n - x^* \rangle = \limsup_{k\to\infty} \langle (\rho V - \mu F) x^*, x_{n_k} - x^* \rangle.$$

Let $x_{n_k} \rightharpoonup w$, as $k \to \infty$. Opial condition guarantees that $x_n \rightharpoonup w$ and we know from Step 1 that $w \in \mathcal{F}$, hence

$$\limsup_{n\to\infty} \langle (\rho V - \mu F) x^*, x_n - x^* \rangle = \langle (\rho V - \mu F) x^*, w - x^* \rangle \leq 0.$$

Step 3. Finally, we show that the sequence $\{x_n\}$ generated by (3.1) converges strongly to the point x^* which is the unique solution of the variational inequality (3.26). From the definition of the iterative sequence $\{x_n\}$, we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \langle P_C t_n - x^*, x_{n+1} - x^* \rangle \\ &= \langle P_C t_n - t_n, x_{n+1} - x^* \rangle + \langle t_n - x^*, x_{n+1} - x^* \rangle. \end{aligned}$$

Also, using property (3) of the metric projection P_C, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \langle t_n - x^*, x_{n+1}-x^*\rangle \\ &= \alpha_n\rho\langle Vx_n - Vx^*, x_{n+1}-x^*\rangle + \alpha_n\langle \rho Vx^* - \mu Fx^*, x_{n+1}-x^*\rangle \\ &\quad + \langle (I-\alpha_n\mu F)T^n y_n - (I-\alpha_n\mu F)T^n x^*, x_{n+1}-x^*\rangle .\end{aligned}$$

Applying Lemma 2, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \frac{(1-\alpha_n(\nu-\rho\gamma))}{2}\left(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\right) \\ &\quad +\alpha_n\langle \rho Vx^* - \mu Fx^*, x_{n+1}-x^*\rangle \\ &\quad +(1-\alpha_n\nu)\beta_n\|Sx^*-x^*\|\|x_{n+1}-x^*\| \\ &\quad +(1-\alpha_n\nu)a_n\|x_{n+1}-x^*\| .\end{aligned}$$

Thus

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \frac{(1-\alpha_n(\nu-\rho\gamma))}{(1+\alpha_n(\nu-\rho\gamma))}\|x_n-x^*\|^2 \\ &\quad +\frac{2\alpha_n}{(1+\alpha_n(\nu-\rho\gamma))}\langle \rho Vx^*-\mu Fx^*, x_{n+1}-x^*\rangle \\ &\quad +\frac{2\beta_n}{(1+\alpha_n(-\rho\gamma))}\|Sx^*-x^*\|\|x_{n+1}-x^*\| \\ &\quad +\frac{2a_n}{(1+\alpha_n(\nu-\rho\gamma))}\|x_{n+1}-x^*\| \\ &\leq (1-\alpha_n(\nu-\rho\gamma))\|x_n-x^*\|^2+\alpha_n(\nu-\rho\gamma)\theta_n\end{aligned}$$

where

$$\theta_n = \frac{2}{(1+\alpha_n(\nu-\rho\gamma))(\nu-\rho\gamma)}\left[\begin{array}{c}\langle \rho Vx^*-\mu Fx^*, x_{n+1}-x^*\rangle + \frac{\beta_n}{\alpha_n}M_3 \\ +\frac{a_n}{\alpha_n}\|x_{n+1}-x^*\|\end{array}\right],$$

and

$$\sup_{n\geq 1}\{\|Sx^*-x^*\|\|x_{n+1}-x^*\|\}\leq M_3.$$

Since $\frac{\beta_n}{\alpha_n}\to 0$ and $\frac{a_n}{\alpha_n}\to 0$, we obtain $\limsup_{n\to\infty}\theta_n\leq 0$. So, it follows from Lemma 3 that the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x^*\in\mathcal{F}$ which is the unique solution of variational inequality (3.26). □

Remark 1. *Our results can be reduced to some corresponding results in the following ways: In iterative process (3.1)*

(1) *if T is a nonexpansive self mapping on C, $G=\varphi=B=A=\beta_n=0$ and $S=I$ for all $x,y\in C$ and $n\in\mathbb{N}$, then we derive the iterative process (1.4) of Ceng et. al. [3] and, in turn, of Moudafi [7] and Marino and Xu [6].*
(2) *if $\beta_n=1$ for all n, then we derive Tian's iterative process [12]*
(3) *if S and T are nonexpansive self mapping and $G=\varphi=B=A=0$, then our process generalizes (1.6)of Wang and Xu. [13] and that of [5].*

(4) *If* $\rho = \mu = 1, V = f, G = \varphi = B = A = 0, F = I$ *and* S *is a self mapping, then we derive iterative process of [16].*

Remark 2. *Since the problem of finding the solution of variational inequality (3.26) is equivalent to finding the solutions of hierarchical fixed point problem*

$$\langle (I - S)\, x^*, x^* - x\rangle \leq 0, \forall x \in \mathcal{F},$$

where $S = I - (\rho V - \mu F)$, *the sequence* $\{x_n\}$ *generated by (3.1) converges strongly to the solutions of this hierarchical fixed point problem.*

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